Rutgers University: Algebra Written Qualifying Exam
August 2015: Problem 4 Solution

Exercise. Let G be a group of order $2015 = 5 \cdot 13 \cdot 31$.

(a) Prove the existence of normal subgroups G of orders 13, 31, and 155.
Hint: establish the existince of those subgroups in that order.

Solution.

Using the Sylow theorems:

$$n_{13} \equiv 1 \mod 13 \qquad \text{and} \qquad n_{13} \mid 155 \qquad \Longrightarrow \qquad n_{13} = 1$$

$\Longrightarrow$ There is one Sylow 13-subgroup, which has order 13, and it is normal in G.

$$n_{31} \equiv 1 \mod 31 \qquad \text{and} \qquad n_{31} \mid 65 \qquad \Longrightarrow \qquad n_{31} = 1$$

$\Longrightarrow$ There is one Sylow 31-subgroup, which has order 31, and it is normal in G.
Let's call this subgroup H_{31}
H_{31} is a cyclic subgroup of order 31 that contains all 30 elements of 31.
Let H_5 denote a Sylow 5-subgroup.
Since 155 is composite, we cannot use typical strategies...

Idea:

(1) Identify a homomorphism from H_5 to the automorphisms of H_{31}
(2) Show normalizer of H_5 in$G$ contains both H_5 and H_{31}
(3) Get $H_5 \lhd G$
(4) Then $H_5 H_{31}$ is a normal subgroup of order 155.

(1) Define homomorphism $f : H_5 \to A(H_{31})$ that sends element $x \in H_5$ to the automorphism xtx^{-1} of H_{31}, $(t \in H_{31})$
(2) If $\phi : G \to H$ and $\gcd(|G|, |H|) = 1$, then ϕ is the trivial homomorphism.
Since the target has order 31 and the source has order 5 and $\gcd(5, 31) = 1$,
f is the trivial homomorphism and so $txt^{-1} = x$ and H_{31} normalizes H_5.
So the normalizer of H_5 in G contains both H_5 and H_{31}.
$\Longrightarrow$ it has at least 155 elements and index of at most 13 by Lagrange.
(3) But, since n_p is the index of the Sylow p-subgroup, its index is $n_5 = 1$ or 31
$\Longrightarrow \mathrm{n}_5 = 1 \implies H_5 \lhd G$
(4) $\implies K = H_5 H_{31}$ is a subgroup with order 155 since $H_5 \cap H_{31} = \{e\}$
Since H_5 and H_{31} are both normal, $\forall g \in G$

$$gKg^{-1} = gH_5H_{31}g^{-1} = gH_5g^{-1}gH_{31}g^{-1} = H_5H_{31} = K$$

$\Longrightarrow K \lhd G$

(b) Show that G is isomorphic to the direct product of a group of order 13 with a group of order 155.

Solution.

H_{13} and K from part (a) are normal subgroups and

$$H_{13} \cap K = \{e\}$$

because the order of the elements divides the order of the subgroup.
$\implies H_{13}K = G$, since $H_{13}K \subset G$ and $|H_{13}K| = \frac{|H_{13}|\cdot|K|}{1} = 13 \cdot 155 = |G|$
It G is a group and H, K are subgroups, and if $G = HK$ then $G \approx H \times K$
$\implies$ G$\approx H_{13} \times K$
Thus, G is isomorphic to the direct product of a group of order 13 and a group of order 155.